

# Single-channel fits and K-matrix constraints

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## Abstract

A K-matrix formalism is used to relate single-channel and multi-channel fits. We show how the single-channel formalism changes as new hadronic channels become accessible. These relations are compared to those commonly used to fit pseudoscalar meson photoproduction data.

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## I. INTRODUCTION AND MOTIVATION

The properties of baryon resonances have largely been obtained from fits to  $\pi N$  elastic scattering and photoproduction data. These fits have covered the range from simple isobar models to more complex analyses involving dispersion-relation constraints. Results have varied widely, particularly for resonances immersed in large backgrounds and coupled only weakly to the initial or final states.

With the availability of high quality data involving final states such as  $\eta N$ ,  $K\Lambda$ , and  $\pi\pi N$ , there have been attempts to fit all connected reactions simultaneously, using unitarity as a constraint. Other groups have concentrated on single-channel fits to data from reactions such as the electro- and photoproduction of etas and kaons. The resulting collections of masses, widths and branching fractions differ significantly in many cases, which has led to debates over the relative importance of particular constraints and approximations.

In the following, we examine the constraints imposed by unitarity on single-channel fits to data using a K-matrix formalism. Pion photoproduction will be illustrated in detail, though our results are easily generalized to other reactions. In pion photoproduction, the constraint imposed by Watson's theorem [1] has been widely used to study the  $\Delta(1232)$  resonance region. For energies above the two-pion production threshold, however, this constraint no longer applies. In an earlier work [2], we outlined a way to extend Watson's theorem beyond the two-pion production threshold, assuming the dominance of a single additional  $\pi\Delta$  channel. This has been used in a number of subsequent fits to pion photoproduction data. In the next section, we recall how the form was derived and show how it reduces to Watson's theorem for an elastic  $\pi N$  scattering amplitude. We then extend this method to account for an arbitrary number of hadronic channels.

It is hoped that this collection of results will allow a more direct comparison between single-channel fits to data and multi-channel analyses based on the K-matrix formalism.

## II. RESULTS WITH ONE AND TWO HADRONIC CHANNELS

The simplest application of a K-matrix constraint to the two-channel process is Watson's theorem, which involves just the channels  $\pi N$  and  $\gamma N$ , showing that below the two-pion production threshold, pion photoproduction amplitudes carry the phase of the associated

$\pi N$  amplitude. In Ref. [2] this argument was extended to include two hadronic channels. The second hadronic channel was labeled  $\pi\Delta$  but was intended to account for all inelasticity in the  $\pi N$  interaction. The  $2 \times 2$  hadronic K- and T-matrices, in this case, were written as

$$K_H = \begin{pmatrix} K_{\pi\pi} & K_{\pi\Delta} \\ K_{\pi\Delta} & K_{\Delta\Delta} \end{pmatrix} \quad (1)$$

and

$$T_H = \begin{pmatrix} T_{\pi\pi} & T_{\pi\Delta} \\ T_{\pi\Delta} & T_{\Delta\Delta} \end{pmatrix} = K_H(1 - iK_H)^{-1}, \quad (2)$$

with the abbreviations  $K(\pi N \rightarrow \pi N) \equiv K_{\pi\pi}$ ,  $K(\pi N \rightarrow \pi\Delta) \equiv K_{\pi\Delta}$ , and  $K(\pi\Delta \rightarrow \pi\Delta) \equiv K_{\Delta\Delta}$ .

Inverting the  $2 \times 2$  hadronic matrix  $(1 - iK_H)$  and multiplying by  $K_H$ , we obtain expressions for the T-matrix elements. The T-matrix element  $T_{\pi\pi}$  can be expressed, in terms of a function  $\bar{K}$ , as

$$T_{\pi\pi} = \frac{\bar{K}}{1 - i\bar{K}} \quad (3)$$

where

$$\bar{K} = K_{\pi\pi} + \frac{iK_{\pi\Delta}^2}{1 - iK_{\Delta\Delta}} \quad (4)$$

and

$$T_{\pi\Delta} = \frac{K_{\pi\Delta}}{1 - iK_{\Delta\Delta}} (1 + iT_{\pi\pi}). \quad (5)$$

The last expression obtains a factor of  $(1 + iT_{\pi\pi})$  through the use of Eq.(3).

Expanding to a  $3 \times 3$  matrix, including the photon interaction channels, we have

$$K = \begin{pmatrix} K_{\gamma\gamma} & K_{\gamma\pi} & K_{\gamma\Delta} \\ K_{\gamma\pi} & K_{\pi\pi} & K_{\pi\Delta} \\ K_{\gamma\Delta} & K_{\pi\Delta} & K_{\Delta\Delta} \end{pmatrix}. \quad (6)$$

Rewriting the relation  $T = K(1 - iK)^{-1}$  for this enlarged K-matrix in the form  $T(1 - iK) = K$  yields a set of relations between the T- and K-matrix elements, a particularly useful one being,

$$T_{\gamma\pi}(1 - iK_{\gamma\gamma}) = (1 + iT_{\pi\pi})K_{\gamma\pi} + iK_{\gamma\Delta}T_{\pi\Delta}. \quad (7)$$

Using the above relation for  $T_{\pi\Delta}$  and keeping terms of first order in the electromagnetic coupling (dropping  $K_{\gamma\gamma}$ ) we have

$$T_{\gamma\pi} = (1 + iT_{\pi\pi}) \left( K_{\gamma\pi} + i \frac{K_{\gamma\Delta}}{K_{\pi\Delta}} \frac{K_{\pi\Delta}^2}{1 - iK_{\Delta\Delta}} \right). \quad (8)$$

Writing this in terms of  $\bar{K}$  and using  $\bar{K}(1 + iT_{\pi\pi}) = T_{\pi\pi}$ , the final result is

$$T_{\gamma\pi} = (1 + iT_{\pi\pi}) \left( K_{\gamma\pi} - \frac{K_{\gamma\Delta} K_{\pi\pi}}{K_{\pi\Delta}} \right) + \frac{K_{\gamma\Delta}}{K_{\pi\Delta}} T_{\pi\pi}. \quad (9)$$

Had we proceeded with only  $\gamma N$  and  $\pi N$  channels, we would have arrived at the simpler expression

$$T_{\gamma\pi} = (1 + iT_{\pi\pi}) K_{\gamma\pi} \quad (10)$$

which, since  $T_{\pi\pi}$  is now elastic, leads directly to Watson's theorem. For real K-matrix elements, Eq.(9) has a phase behavior related to  $T_{\pi\pi}$  and provides a smooth connection of Watson's theorem to energies above the two-pion production threshold.

### III. EXTENSION TO THREE OR MORE HADRONIC CHANNELS

The extension to a third hadronic channel, such as  $\eta N$ , can be handled using the method described above. Writing  $T_H(1 - iK_H) = K_H$  for a 3-channel K-matrix, involving the elements  $\pi N$ ,  $\pi\Delta$  and  $\eta N$ , again yields useful relations. In order to arrive at an expression similar to Eq.(9), we retain combinations containing  $T_{\pi\pi}$

$$T_{\pi\Delta} = \frac{K_{\pi\Delta}}{1 - iK_{\Delta\Delta}} (1 + iT_{\pi\pi}) + \frac{iK_{\Delta\eta}}{1 - iK_{\Delta\Delta}} T_{\pi\eta} \quad (11)$$

and

$$T_{\pi\eta} = \frac{1}{D} \left( K_{\pi\eta} + \frac{iK_{\pi\Delta} K_{\Delta\eta}}{1 - iK_{\Delta\Delta}} \right) (1 + iT_{\pi\pi}) \quad (12)$$

with

$$D = 1 - iK_{\eta\eta} + K_{\Delta\eta}^2 / (1 - iK_{\Delta\Delta}). \quad (13)$$

The  $\pi N$  T-matrix element can again be represented in terms of a function  $\bar{K}$ , as in Eq.(3), with

$$\bar{K} = K_{\pi\pi} + \frac{iK_{\pi\Delta}^2}{1 - iK_{\Delta\Delta}} - \frac{2}{D} \frac{K_{\pi\Delta} K_{\pi\eta} K_{\Delta\eta}}{1 - iK_{\Delta\Delta}} - \frac{i}{D} \left( \frac{K_{\pi\Delta} K_{\Delta\eta}}{1 - iK_{\Delta\Delta}} \right)^2 + \frac{i}{D} K_{\pi\eta}^2 \quad (14)$$

Substituting these relations into

$$T_{\gamma\pi}(1 - iK_{\gamma\gamma}) = (1 + iT_{\pi\pi}) K_{\gamma\pi} + iK_{\gamma\Delta} T_{\pi\Delta} + iK_{\gamma\eta} T_{\pi\eta} \quad (15)$$

dropping the  $K_{\gamma\gamma}$  term and regrouping, we have

$$T_{\gamma\pi} = (1 + iT_{\pi\pi}) \left( K_{\gamma\pi} - \frac{K_{\gamma\Delta} K_{\pi\pi}}{K_{\pi\Delta}} \right) + (1 + iT_{\pi\pi}) \left( A \frac{K_{\gamma\Delta}}{K_{\pi\Delta}} + B \frac{K_{\gamma\eta}}{K_{\pi\eta}} \right) \quad (16)$$

with

$$A = K_{\pi\pi} + \frac{iK_{\pi\Delta}^2}{1 - iK_{\Delta\Delta}} - \frac{1}{D} \frac{K_{\pi\Delta}K_{\pi\eta}K_{\Delta\eta}}{1 - iK_{\Delta\Delta}} - \frac{i}{D} \left( \frac{K_{\pi\Delta}K_{\Delta\eta}}{1 - iK_{\Delta\Delta}} \right)^2 \quad (17)$$

and

$$B = \frac{i}{D} \left( K_{\pi\eta}^2 + \frac{iK_{\pi\Delta}K_{\pi\eta}K_{\Delta\eta}}{1 - iK_{\Delta\Delta}} \right) \quad (18)$$

Since  $A + B = \bar{K}$ , we can add and subtract  $BK_{\gamma\Delta}/K_{\pi\Delta}$  and use  $\bar{K}(1 + iT_{\pi\pi}) = T_{\pi\pi}$  to obtain

$$T_{\gamma\pi} = (1 + iT_{\pi\pi}) \left( K_{\gamma\pi} - \frac{K_{\gamma\Delta}K_{\pi\pi}}{K_{\pi\Delta}} \right) + \frac{K_{\gamma\Delta}}{K_{\pi\Delta}} T_{\pi\pi} + i \left( K_{\gamma\eta} - \frac{K_{\gamma\Delta}}{K_{\pi\Delta}} K_{\pi\eta} \right) T_{\pi\eta}. \quad (19)$$

The extension to further hadronic channels is then obvious. Each new channel  $mN$  adds a term of the form

$$i \left( K_{\gamma m} - \frac{K_{\gamma\Delta}}{K_{\pi\Delta}} K_{\pi m} \right) T_{\pi m}. \quad (20)$$

#### IV. RESULTS AT THE K-MATRIX POLE

It is instructive to examine these results at the K-matrix pole position. Taking the simplest representation involving a single pole

$$K_{\gamma\pi} = \frac{A}{W - W_R} + B ; \quad K_{\pi\pi} = \frac{C}{W - W_R} + D \quad (21)$$

in the relation  $T_{\gamma\pi} = K_{\gamma\pi}(1 + iT_{\pi\pi})$ , one obtains

$$T_{\gamma\pi} = \left( \frac{A}{W - W_R} + B \right) \frac{(W - W_R)}{W - W_R - i[C + (W - W_R)D]} \quad (22)$$

Of the two resulting terms, one is clearly ‘resonant’ at  $W = W_R$  while the other goes to zero at this energy (the ‘background’ term). Both have the phase of the (elastic)  $\pi N$  T-matrix.

This nice correspondence disappears when a second hadronic channel is added. With the addition of a  $\pi\Delta$  channel,  $(1 + iT_{\pi\pi})$  no longer goes to zero at  $W = W_R$ , and the expression in Eq.(22) becomes divergent. Assuming a pole exists in  $K_{\gamma\Delta}$ , both terms giving  $T_{\gamma\pi}$  in Eq.(7) diverge, though the sum remains finite. In our Eq.(19), however, these divergences do not occur. The bracketed terms remain finite at the K-matrix pole, assuming the residues are factorizable.

In the simple case (Watson’s theorem) involving only the  $\gamma N$  and  $\pi N$  channels, there would have been no resonance term without a pole in the K-matrix element  $K_{\gamma\pi}$ . However,

in Eq.(9) and its generalization, Eq.(19), the pole structures in all K-matrix elements multiplying the hadronic T-matrices cancel by construction. The resonancelike behavior of  $T_{\gamma\pi}$ , in this case, results from the structure of the hadronic T-matrices.

It is amusing to consider a case without explicit K-matrix poles which results in resonance-like behavior. This is most easily seen from Eq.(4), which is in fact the reduced K-matrix (element) of the full K-matrix involving  $\pi N$  and  $\pi\Delta$  channels. As is well known [3], the reduced K-matrix can (in principle) develop a pole, due to a zero in the denominator of the second term in Eq.(4), without an explicit pole in  $K_{\pi\pi}$ . This would typically occur just below the threshold for  $\pi\Delta$  production, where momentum factors implicit in  $K_{\Delta\Delta}$  become complex. However, expressions of the form

$$\left(K_{\gamma\pi} - \frac{K_{\gamma\Delta}}{K_{\pi\Delta}}K_{\pi\pi}\right) \quad (23)$$

remain finite through the cancellation of poles at the resonance position. The behavior of such terms would be completely different if some of the elements were pole free. However, if none of the K-matrix elements contained poles, the result could be similar to one due to pole cancellation.

## V. SINGLE-CHANNEL VERSUS MULTI-CHANNEL FITS

We have shown how the functional forms used in fitting single-channel photoproduction data are related to the K-matrix elements of a multi-channel analysis. In the  $\Delta(1232)$  resonance region, Watson's theorem provides a strong constraint on the fit. If only two hadronic channels dominate, a form involving only the  $\pi N$  T-matrix is also possible. Beyond this point, terms proportional to other hadronic T-matrices appear to be necessary.

It is useful to compare two single-channel forms that have been used extensively in fitting pion photoproduction data. In the original SAID [2] analysis, data were fitted using the relation

$$T_{\gamma\pi} = (\text{Born} + A)(1 + iT_{\pi\pi}) + BT_{\pi\pi} \quad (24)$$

wherein the 'Born' term included vector meson exchanges and the terms  $A$  and  $B$  were phenomenological. This corresponds to Eq.(9), with  $A$  and  $B$  representing the ratios of K-matrix elements. In the original MAID fits [4] a simpler parametrization

$$T_{\gamma\pi} = (\text{Born})(1 + iT_{\pi\pi}) + e^{i\phi}T_{\text{BW}} \quad (25)$$

was used. This form explicitly separates resonance and background pieces, the resonance part taken to be a Breit-Wigner function. The parameter  $\phi$  is included to ensure that the overall phases of  $T_{\gamma\pi}$  and  $T_{\pi\pi}$  are equal at energies where Watson's theorem is valid.

Differences between Eqs.(24) and (25) are minimized near resonances that have small backgrounds, a clear Breit-Wigner behavior, and a large coupling to the  $\pi N$  channel. For a nearly elastic resonance, the term  $(1 + iT_{\pi\pi})$  becomes small and masks the effect of the additional term  $A$  in Eq.(24). It should be noted, however, that many established resonances have  $\pi N$  branching fractions of only 10-30% . In an unbiased fit, the term  $A$  can become quite large, resulting in a 'background' very different from the first term in Eq.(25). An example of this effect will be given below.

In order to make contact with multi-channel fits, comparisons should be made in cases where the dominance of two hadronic channels is a good approximation. Here the quantity  $A$ , found from fits to single-channel data, could be directly compared to calculated ratios of K-matrix elements. One problem with this approach is the reliable determination of  $K_{\gamma\pi}$ . Missing pieces in  $K_{\gamma\pi}$  could be compensated for in the phenomenological term. A consistent comparison would require the same form of  $K_{\gamma\pi}$  to be used in both the single- and multi-channel analyses.

Finally, we mention a reaction, other than pion photoproduction, where the dominance of two hadronic channels is a viable approximation. Rewriting Eq.(9) for eta photoproduction in the N(1535) channel, we have

$$T_{\gamma\eta} = (1 + iT_{\eta\eta}) \left( K_{\gamma\eta} - \frac{K_{\gamma\pi}K_{\eta\eta}}{K_{\pi\eta}} \right) + \frac{K_{\gamma\pi}}{K_{\pi\eta}} T_{\eta\eta} \quad (26)$$

assuming the dominance of  $\pi N$  and  $\eta N$ . Early single-channel fits assumed eta photoproduction was resonance dominated with negligible background. The argument for this was based on the size of Born contributions involving a very small  $\eta NN$  coupling constant. However, while small Born terms may justify the neglect of  $K_{\gamma\eta}$ , this argument does not exclude the second term multiplying  $(1 + iT_{\eta\eta})$ . This gives a qualitative way to understand the different results of resonance-only single-channel fits [5] and multichannel analyses [6].

## VI. A NUMERICAL COMPARISON

In Fig. 1, we compare a Born-term approximation to the sum of Born term and phenomenological contributions multiplying  $(1 + iT_{\pi\pi})$  in Eq. (24). The phenomenological term is parameterized as a polynomial in energy, constrained to have the proper threshold behavior. Figure 1(a) shows the multipole  $M_{1-}^{1/2}$  connected to the Roper resonance, which couples largely to  $\pi N$  and  $\pi\Delta$ . Figure 1(b) gives the magnetic multipole connected to the  $D_{13}(1520)$   $\pi N$  resonance. This state couples largely to  $\pi N$  and  $\pi\Delta$ , but also has a substantial  $\rho N$  coupling. In an unbiased fit to data, the factors multiplying  $(1 + iT_{\pi\pi})$  rapidly depart from a simple Born term approximation.

The use of Born terms with point-like couplings is known to be problematic both at threshold (where chiral perturbation theory is applicable) and at higher energies (where form factors are often applied). With this in mind, the departure from a Born approximation to  $K_{\gamma\pi}$  should not be surprising. Notice that the sum of Born and phenomenological pieces changes sign near the resonance position in these two multipoles. This behavior is more likely due to the phenomenological term than a modification of the Born approximation. While the pole terms multiplying hadronic T-matrices in Eq. (9) cancel at  $W_R$ , non-pole cross terms retain a dependence on  $(W - W_R)$ . These terms could account for a sign change near the resonance energy. Not all multipoles show this cross-over behavior. The evaluation of  $K_{\gamma\Delta}K_{\pi\pi}/K_{\pi\Delta}$  within a multi-channel model could help to clarify this issue. It should be emphasized that the phenomenological terms used to generate the curves in Fig. 1 contain no explicit dependence on  $W_R$ .

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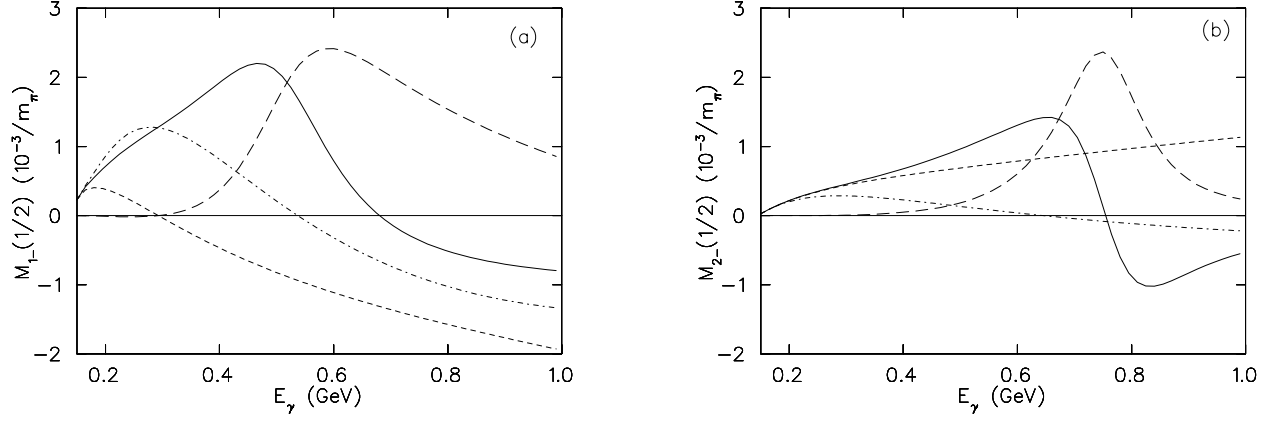


FIG. 1: (a)  $M_{1-}^{1/2}$  and (b)  $M_{2-}^{1/2}$  multipole amplitudes. Solid (long dashed) curves give the real (imaginary) parts of the SM95 [7] multipoles. The short-dashed curves give the (real) Born terms and the dot-dashed curves give the sum of Born + phenomenological terms.

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